Locally homogeneous rigid geometric structures on surfaces

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ABSTRACT. We study locally homogeneous rigid geometric structures on surfaces. We show that a locally homogeneous projective connection on a compact surface is flat. We also show that a locally homogeneous unimodular affine connection ∇ on a two dimensional torus is complete and, up to a finite cover, homogeneous.

Let ∇ be a unimodular real analytic affine connection on a real analytic compact connected surface M. If ∇ is locally homogeneous on a nontrivial open set in M, we prove that ∇ is locally homogeneous on all of M.

1. Introduction

Riemannian metrics are the most common (rigid) geometric structures. A locally homogeneous Riemannian metric on a surface has constant sectional curvature and it is locally isometric either to the standard metric on the two-sphere, or to the flat metric on $\mathbb{R}^2$, or to the hyperbolic metric on the Poincaré's upper-half plane (hyperbolic plane). Obviously the flat metric is translation invariant on $\mathbb{R}^2$. Recall also that the isometry group of the hyperbolic plane is isomorphic to $PSL(2, \mathbb{R})$ and contains copies of the affine group of the real line (preserving orientation) $Aff(\mathbb{R})$, which act freely and transitively (they are the stabilizers of points on the boundary). Consequently, the hyperbolic plane is locally isometric to a translation invariant Riemannian metric on $Aff(\mathbb{R})$.

We generalize here this phenomenon to all rigid geometric structures in Gromov's sense (see the definition in the following section).

Theorem 1.1. *Let ϕ be a locally homogeneous rigid geometric structure on a surface. Then ϕ is locally isomorphic to a rigid geometric structure which is either rotation invariant on the two-sphere, or translation invariant on $\mathbb{R}^2$, or translation invariant on the affine group of the real line preserving orientation $Aff(\mathbb{R})$.*

If ϕ is an affine connection this was first proved by B. Opozda [32] for the torsion-free case (see also the group-theoretical approach in [22]), and then by T. Arias-Marco and O. Kowalski in the case of arbitrary torsion [3]. Nevertheless theorem 1.1 follows directly from a theorem of Mostow and

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from Lie's classification of two-dimensional homogeneous spaces and seems to be known in the field.

Theorem 1.1 stands, in particular, for *projective connections*. Recall that a projective connection is a class of affine connections, two affine connections being equivalent if they have the same (unparametrized) geodesics.

In this case the theorem can also be deduced from the results obtained in [9]. Recall that a well-known class of locally homogeneous projective connections ϕ on surfaces are those which are *flat*, i.e. locally isomorphic to the standard projective connection of the projective plane $P^2(\mathbb{R})$. The automorphism group of $P^2(\mathbb{R})$ is the projective group $PGL(3, \mathbb{R})$ which contains $\mathbb{R}^2$ acting freely and transitively (by translation) on the affine plane: it is the subgroup which fixes each point in the line at the infinity. Consequently, ϕ is locally isomorphic to a translation invariant projective connection on $\mathbb{R}^2$.

We prove also the following global results dealing with projective and affine connections on surfaces:

Theorem 1.2. *A locally homogeneous projective connection on a compact surface is flat.*

Theorem 1.3. *Let M be a compact connected real analytic surface endowed with a unimodular real analytic affine connection ∇. If ∇ is locally homogeneous on a nontrivial open set in M, then ∇ is locally homogeneous on M.*

The main result of the article is Theorem 1.3. It is motivated by the celebrated open-dense orbit theorem of M. Gromov [13, 18] (see also [5, 11, 16]). Gromov's result asserts that a rigid geometric structure admitting an automorphism group which acts with a dense orbit is locally homogeneous on an open dense set. This maximal locally homogeneous open (dense) set appears to be mysterious and it might very well happen that it coincides with all of the (connected) manifold in many interesting geometric backgrounds. This was proved, for instance, for Anosov flows with differentiable stable and instable foliations and transverse contact structure [6] and for three dimensional compact Lorentz manifolds admitting a nonproper one parameter group acting by automorphisms [36]. In [7], the authors deal with this question and their results indicate ways in which the rigid geometric structure cannot degenerate off the open dense set.

Surprisingly, the extension of a locally homogeneous open dense subset to all of the (connected) manifold might stand *even without assuming the existence of a big automorphism group*. This is known to be true in the Riemannian setting [17], as a consequence of the fact that all scalar invariants are constant (see also Corollary 4.2 in our section 4). This was also recently proved in the frame of three dimensional real analytic Lorentz metrics [14] and, in higher dimesnion, for complete real analytic Lorentz metrics having semisimple Killing Lie algebra [25] (in the last paper the conclusion is stronger: the local isometry orbits are, roughly, fibers of a fiber bundle).

Theorem 1.3 proves the extension phenomenon in the setting of unimodular real analytic affine connections on compact surfaces. This result is no longer true when the manifold is noncompact (see the example given in proposition 4.18).

As a by-product of the proof we get the following:

Theorem 1.4. *A locally homogeneous unimodular affine connection on a two dimensional torus is complete and, up to a finite cover, homogeneous.*

The composition of the article is the following. In section 2 we introduce the basic facts about rigid geometric structures and prove theorem 1.1. Section 3 deals with global phenomena and proves theorem 1.2. Theorems 1.3 and 1.4 will be proved in sections 4 and 5 respectively.

I would like to thank Adolfo Guillot who helped me to fill a gap in the proof of theorem 1.3 in a first version of this paper and gave a nice example of a non locally homogeneous unimodular analytic affine connection on $\mathbb{R}^2$ which is locally homogeneous on nontrivial open sets, but not on all of $\mathbb{R}^2$ (see proposition 4.18).

2. Locally homogeneous rigid geometric structures

In the sequel all manifolds will be supposed to be smooth and connected. The geometric structures will be also assumed to be smooth.

Consider a n-manifold M and, for all integers $r \geq 1$, consider the associated bundle $R^r(M)$ of r-frames, which is a $D^r(\mathbb{R}^n)$-principal bundle over M, with $D^r(\mathbb{R}^n)$ the real algebraic group of r-jets at the origin of local diffeomorphisms of $\mathbb{R}^n$ fixing 0 (see [1]).

Let us consider, as in [13, 18], the following

Definition 2.1. A *geometric structure* (of order r and of algebraic type) ϕ on a manifold M is a $D^r(\mathbb{R}^n)$-equivariant smooth map from $R^r(M)$ to a real algebraic variety Z endowed with an algebraic action of $D^r(\mathbb{R}^n)$.

Riemannian and pseudo-Riemannian metrics, affine and projective connections and the most encountered geometric objects in differential geometry are known to verify the previous definition [13, 18, 5, 11, 16]. For instance, if the image of ϕ in Z is exactly one orbit, this orbit identifies with a homogeneous space $D^r(\mathbb{R}^n)/G$, where G is the stabilizer of a chosen point in the image of ϕ. We get then a reduction of the structure group of $R^r(M)$ to the subgroup G. This is exactly the classical definition of a G-structure (of order r): the case $r = 1$ and $G = O(n, \mathbb{R})$ corresponds to a Riemannian metric and that of $r = 2$ and $G = GL(n, \mathbb{R})$ gives a torsion free affine connection [20, 1].

Definition 2.2. A (local) Killing field of ϕ is a (local) vector field on M whose canonical lift to $R^r(M)$ preserves ϕ.

Following Gromov [18, 13] we define rigidity as:

Definition 2.3. A geometric structure ϕ is rigid at order $k \in \mathbb{N}$, if any isometric jet of order $k + 1$ is completely determined by the underlying k-order jet.

Consequently, in the neighborhood of any point of M, the algebra of Killing fields of a rigid geometric structure is finite dimensional.

Recall that (pseudo)-Riemannian metrics, as well as affine and projective connections, or conformal structures in dimension ≥ 3 are known to be rigid [13, 18, 5, 11, 16, 20].

Definition 2.4. The geometric structure ϕ is said to be locally homogeneous on the open subset $U \subset M$ if for any tangent vector $V \in T_uU$ there exists a local Killing field X of ϕ such that $X(u) = V$.

The Lie algebra of Killing fields is the same at the neighborhood of any point of a locally homogeneous geometric structure ϕ. In this case it will be simply called *the Killing algebra of* ϕ.

Let G be a connected Lie group and I a closed subgroup of G. Recall that M is said to be *locally modelled* on the homogeneous space G/I if it admits an atlas with open sets diffeomorphic to open sets in G/I such that the transition maps are given by restrictions of elements in G on each component.

In this situation any G-invariant geometric structure $\tilde{\phi}$ on G/I uniquely defines a locally homogeneous geometric structure ϕ on M which is locally isomorphic to $\tilde{\phi}$.

We recall that there exists locally homogeneous Riemannian metrics on 5-dimensional manifolds which are not locally isometric to a invariant Riemannian metric on a homogeneous space [21, 23]. However this phenomenon cannot happen in lower dimension:

Theorem 2.5. *Let M be a manifold of dimension ≤ 4 bearing a locally homogeneous rigid geometric structure ϕ with Killing algebra $\mathfrak{g}$. Then M is locally modelled on a homogeneous space G/I, where G is a connected Lie group with Lie algebra $\mathfrak{g}$ and I is a closed subgroup of G. Moreover, (M, ϕ) is locally isomorphic to a G-invariant geometric structure on G/I.*

Proof. Let $\mathfrak{g}$ be the Killing algebra of ϕ. Denote by $\mathfrak{I}$ the (isotropy) subalgebra of $\mathfrak{g}$ composed by Killing fields vanishing at a given point in M. Let G be the unique connected simply connected Lie group with Lie algebra $\mathfrak{g}$. Since $\mathfrak{I}$ is of codimension ≤ 4 in $\mathfrak{g}$, a result of Mostow [28] (chapter 5, page 614) shows that the Lie subgroup I in G associated to $\mathfrak{I}$ is *closed*. Then ϕ induces a G-invariant geometric structure $\tilde{\phi}$ on G/I locally isomorphic to it. Moreover, M is locally modelled on G/I. $\square$

Remark 2.6. *By the previous construction, G is simply connected and I is connected (and closed), which implies that G/I is simply connected (see* [28]*, page 617, Corollary 1). In general, the G-action on G/I admits a nontrivial discrete kernel. We can assume that this action is effective considering the quotient of G and I by the maximal normal subgroup of G contained in I (see proposition 3.1 in* [34]*).*

Theorem 1.1 is a direct consequence of theorem 2.5 and of the following:

Proposition 2.7. *A two dimensional homogeneous space G/I of a connected simply connected Lie group G is either the two sphere with G being S^3, or it bears an action of a two dimensional subgroup of G (isomorphic either to $\mathbb{R}^2$, or to the affine group of the real line preserving orientation) which admits an open orbit.*

Here the 3-sphere S^3 is endowed with its standard structure of Lie group [19, 31].

Proof. This follows directly from Lie's classification of the two dimensional homogeneous space (see the list in [31] or [28]). More precisely, any (finite dimensional) Lie algebra acting transitively on a surface either admits a two dimensional subalgebra acting simply transitively (or equivalently, which trivially intersects the isotropy subalgebra), or it is isomorphic to the Lie algebra of S^3 acting on the real sphere S^2 by the standard action. $\square$

3. Global rigidity results

Recall that a manifold M locally modelled on a homogeneous space G/I gives rise to a *developing map* defined on its universal cover $\tilde{M}$ with values in G/I and to a *holonomy morphism* $\rho : \pi_1(M) \to G$ (well defined up to conjugacy in G) [34]. The developing map is a local diffeomorphism which is equivariant with respect to the action of the fundamental group $\pi_1(M)$ on $\tilde{M}$ (by deck transformations) and on G (by its image through ρ).

The manifold M is said to be *complete* if the developing map is a global diffeomorphism. In this case M is diffeomorphic to a quotient of G/I by a discrete subgroup of G acting properly and without fixed points.

We give now a last definition:

Definition 3.1. A geometric structure ϕ on M is said to be of *Riemannian type* if there exists a Riemannian metric on M preserved by all Killing fields of ϕ.

Roughly speaking a locally homogeneous geometric structure is of Riemannian type if it is constructed by putting together a Riemannian metric and any other geometric structure (e.g. a vector field). Since Riemannian metrics are rigid, a geometric structure of Riemannian type it is automaticaly rigid.

With this terminology we have the following corollary of theorem 2.5.

Theorem 3.2. *Let M be a compact manifold of dimension ≤ 4 equipped with a locally homogeneous geometric structure ϕ of Riemannian type. Then M is isomorphic to a quotient of a homogeneous space G/I, endowed with a G-invariant geometric structure, by a lattice in G.*

Proof. By theorem 2.5, M is locally modelled on a homogeneous space G/I. Since ϕ is of Riemannian type, G/I admits a G-invariant Riemannian metric. This implies that the isotropy I is compact.

On the other hand, compact manifolds locally modelled on homogeneous space G/I with compact isotropy group I are classically known to be complete: this is a consequence of the Hopf-Rinow's geodesical completeness [34]. $\square$

Remark 3.3. *A G-invariant geometric structure on G/I is of Riemannian type if and only if I is compact.*

Recall that a homogeneous space G/I is said to be *imprimitive* if the canonical G-action preserves a non trivial foliation.

Proposition 3.4. *If M is a compact surface locally modelled on an imprimitive homogeneous space, then M is a torus.*

Proof. The G-invariant one dimensional foliation on G/H descends on M to a non singular foliation. Hopf-Poincaré's theorem implies then that the genus of M equals one: M is a torus. $\square$

Note that the results of [22, 3] imply in particular:

Theorem 3.5. *A locally homogeneous affine connection on a surface which is neither torsion free and flat, nor of Riemannian type, is locally modelled on an imprimitive homogeneous space.*

Indeed, T. Arias-Marco and O. Kowalski study in [3] all possible local normal forms for locally homogeneous affine connections on surfaces with the corresponding Killing algebra. Their results are summarized in a nice table (see [3], pages 3-5). In all cases, except for the Killing algebra of the (standard) torsion free affine connection and for Levi-Civita connections of Riemannian metrics of constant sectional curvature, there exists at least one Killing field non contained in the isotropy algebra which is normalized by the Killing algebra. Its direction defines then a G-invariant line field on G/I.

The previous result combined with proposition 3.4 imply the main result in [32]:

Theorem 3.6. *(Opozda) A compact surface M bearing a locally homogeneous affine connection of non Riemannian type is a torus.*

Recall first that a well known result of J. Milnor shows that a compact surface bearing a flat affine connection is a torus [26] (see also [8]).

Proof. In the case of a non flat connection, theorem 3.5 shows that M is locally modelled on an imprimitive homogeneous space. Then proposition 3.4 finishes the proof. $\square$

We give now the proof of theorem 1.2.

Proof. The starting point of the proof is the classification obtained in [9] of all possible Killing algebras of a two dimensional locally homogeneous projective connection. Indeed, Lemma 3 and Lemma 4 in [9] prove that eiher ϕ is flat, or the Killing algebra of ϕ is the Lie algebra of one of the following Lie groups: $Aff(\mathbb{R})$ or $SL(2,\mathbb{R})$. Moreover, in the last case the isotropy is generated by a one parameter unipotent subgroup.

Assume, by contradiction, that the Killing algebra of ϕ is that of $Aff(\mathbb{R})$. Then, by theorem 2.5, M is locally modelled on $Aff(\mathbb{R})$ and, by theorem 3.2, M has to be a quotient of $Aff(\mathbb{R})$ by a uniform lattice. Or, $Aff(\mathbb{R})$ is not unimodular and, consequently, doesn't admit lattices: a contradicition.

Assume, by contradiction, that the Killing algebra is that of $SL(2,\mathbb{R})$. Then, by theorem 2.5, M is locally modelled on $SL(2,\mathbb{R})/I$, with I a one parameter unipotent subgroup in $SL(2,\mathbb{R})$.

Equivalently, I is conjugated to $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, with $b \in \mathbb{R}$. The homogeneous space $SL(2,\mathbb{R})/I$ is difffeomorphic to $\mathbb{R}^2 \setminus \{0\}$ endowed with the linear action of $SL(2,\mathbb{R})$.

Notice that the action of $SL(2,\mathbb{R})$ on $SL(2,\mathbb{R})/I$ preserves a nontrivial vector field. The expression of this vector field in linear coordinates (x_1, x_2)

on $\mathbb{R}^2 \setminus \{0\}$ is $x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$, which is the fundamental generator of the one parameter group of homotheties. The flow of this vector field doesn't preserve the standard volume form and has a nonzero constant divergence: $\lambda = 2$.

Let X be the corresponding vector field induced on M and $div(X)$ the divergence of X with respect to the volume form vol induced on M by the standard $SL(2,\mathbb{R})$-invariant volume form of $\mathbb{R}^2 \setminus \{0\}$. Recall that, by definition, $L_X vol = div(X) \cdot vol$, where L_X is the Lie derivative. Here $div(X)$ is the constant function λ.

Denote by Ψ^t the time t of the flow generated by X. We get $(\Psi^t)^* vol = exp(\lambda t) \cdot vol$, for all $t \in \mathbb{R}$. But the flow of X has to preserve the global volume $\int_M vol$. This implies $\lambda = 0$: a contradiction.

Thus ϕ must be flat. □

We terminate the section with the following.

Proposition 3.7. *If M is a compact surface endowed with a locally homogeneous rigid geometric structure admitting a semi-simple Killing algebra of dimension* 3*, then either M is globally isomorphic to a rotation invariant geometric structure on the two-sphere (up to a double cover), or the Killing algebra of ϕ preserves a hyperbolic metric on M (and M is of genus $g \geq 2$).*

Proof. By theorem 2.5, M is locally modelled on G/I, with G a 3-dimensional connected simply connected semi-simple Lie group and I a closed one parameter subgroup in G.

Up to isogeny, there are only two such G: S^3 and $SL(2,\mathbb{R})$ [19]. If $G = S^3$ then I is compact and coincides with the stabilizer of a point under the standard S^3-action on S^2. Consequently, G/I identifies with S^2 and the G-action on G/I preserves the canonical metric of the two-sphere.

The developing map from $\tilde{M}$ to G/I has to be a diffeomorphism (see theorem 3.2). Consequently, M is a quotient of the sphere S^2 by a discrete subgroup of G acting by deck transformations. Since a nontrivial isometry of S^2 which is not $-Id$ always admits fixed points, this discrete subgroup has to be of order at most two. Up tp a double cover, (M, ϕ) is isomorphic to S^2 endowed with a rotation invariant geometric structure.

Consider now the case where $G = SL(2,\mathbb{R})$. Then M is locally modelled on G/I, where I is a closed one parameter subgroup in $SL(2,\mathbb{R})$. We showed in the proof of theorem 1.2 that I is not conjugated to a unipotent subgroup.

Assume now that I is conjugated to a one parameter semi-simple subgroup in $SL(2,\mathbb{R})$. We prove that this assumption yields a contradiction. In order to describe the geometry of $SL(2,\mathbb{R})/I$, consider the adjoint representation of $SL(2,\mathbb{R})$ into its Lie algebra $sl(2,\mathbb{R})$. This $SL(2,\mathbb{R})$-action preserves the Killing quadratic form q, which is a non degenerate Lorentz quadratic form. Choose $x \in sl(2,\mathbb{R})$ a vector of unitary q-norm and consider its orbit under the adjoint representation. This orbit identifies with our homogeneous space $SL(2,\mathbb{R})/I$, on which the restriction of the Killing form induces a two dimensional $SL(2,\mathbb{R})$-invariant complete Lorentz metric g of constant nonzero sectional curvature [35].

But there is no *compact* surface locally modelled on the previous homogeneous space $SL(2,\mathbb{R})/I$: this is a special case of the Calabi-Markus phenomenon.

Thus I must be conjugated to a one parameter subgroup of rotations in $SL(2,\mathbb{R})$ and then the $SL(2,\mathbb{R})$-action on $SL(2,\mathbb{R})/I$ identifies with the action by homographies on the Poincaré's upper-half plane. This action preserves the hyperbolic metric. Therefore M inherits a hyperbolic metric and, consequently, its genus is ≥ 2. □

4. Dynamics of the local Killing algebra

A manifold M bearing a geometric structure ϕ admits a natural partition given by the orbits of the action of the Killing algebra of ϕ. Precisely, two points $m_1, m_2 \in M$ are in the same subset of the partition if m_1 can be reached from m_2 by flowing along a finite sequence of local Killing fields. A connected open set in M where ϕ is locally homogeneous lies in the same subset of this partition.

The Gromov's celebrated stratification theorem [13, 18] which was used and studied by many authors [13, 18, 5, 11, 16] roughly states that, if ϕ is rigid, the subsets of this partition are locally closed in M. When ϕ is of Riemannian type, the following more precise statements are well-known:

Theorem 4.1. *Let M be a connected manifold endowed with a geometric structure of Riemannian kind ϕ. Then the orbits of the Killing algebra of ϕ are closed.*

Corollary 4.2. *If ϕ is locally homogeneous on an open dense set, then ϕ is locally homogeneous on M.*

Recall also that Riemannian metrics all of whose scalar invariants are constant are locally homogeneous [17] (this is known to fail in the pseudo-Riemannian setting [10]).

In the real analytic realm this implies the following more precise:

Corollary 4.3. *If M and ϕ are real analytic and ϕ is locally homogeneous on a nontrivial open set, then ϕ is locally homogeneous on M.*

Proof. In the real analytic setting, Gromov's proof shows that away from a nowhere dense analytic subset S in M, the orbits of the Killing algebra are connected components of fibers of an analytic map of constant rank [18] (section 3.2). With our hypothesis, the orbits of the Killing algebra are exactly connected components of $M \setminus S$. Consequently, they are open sets. One apply now corollary 4.2 and get that the orbits are also closed. Since M is connected, the Killing algebra admits exactly one orbit. □

In the previous results the compactness of the orthogonal group was essential.

We prove now theorem 1.3 which deals with unimodular affine connections which are not (necessarily) of Riemannian kind.

Recall that ∇ is said *unimodular* if there exists a volume form on M which is invariant by the parallel transport [1]. This volume form is automaticaly preserved by any local Killing field of ∇. We prove first the following useful:

Lemma 4.4. *Let ∇ be a unimodular analytic affine connection on an analytic surface M. Then the dimension of the isotropy algebra at a point of M is $\neq 2$.*

Proof. Assume by contradiction that the isotropy algebra $\mathcal{I}$ at a point $m \in M$ has dimension two. Consider a system of local exponential coordinates at m with respect to ∇ and, for all $k \in \mathbb{N}$ take the k-jet of ∇ in these coordinates (see [18, 13]). Any volume preserving linear isomorphism of T_mM gives another system of local exponential coordinates at m, with respect to which we consider the k-jet of the connection. This gives an algebraic $SL(2,\mathbb{R})$-action on the vector space Z^k of k-jets of affine connections on $\mathbb{R}^2$ [13, 18].

Elements of $\mathcal{I}$ linearize in exponential coordinates at m (see, for instance, [35] or [33], section 3.1). The main idea in the proof of the linearization result is that geodesics are integral curves of a vector field in TM with quadratic homogeneous vertical part. In exponential coordinates at m, geodesics passing through m locally rectify on lines with constant speed in T_mM passing through m and any local isometry preserving m will be linear. In exponential coordinates Christoffel symbols satisfy $\Gamma^k_{ij} + \Gamma^k_{ji} = 0$ at m. Moreover, if ∇ is torsion free that implies the vanishing of Christoffel symbols at m.

Since elements of $\mathcal{I}$ preserve ∇, they preserve in particular the k-jet of ∇ at m, for all $k \in \mathbb{N}$. This gives an embedding of $\mathcal{I}$ in the Lie algebra of $SL(2,\mathbb{R})$ such that the corresponding (two dimensional) connected subgroup of $SL(2,\mathbb{R})$ preserves the k-jet of ∇ at m for all $k \in \mathbb{N}$.

Now we use the fact that *the stabilizers of a linear algebraic $SL(2,\mathbb{R})$-action are of dimension $\neq 2$*. Indeed, it suffices to check this statement for irreducible linear representations of $SL(2,\mathbb{R})$ for which it is well-known that the stabilizer in $SL(2,\mathbb{R})$ of a nonzero element is one dimensional [19].

It follows that the stabilizer of the k-jet of ∇ at m is of dimension three and contains the connected component of identity in $SL(2,\mathbb{R})$. Consequently, in exponential coordinates at m, each element of the connected component of the identity in $SL(2,\mathbb{R})$ gives rise to a local linear vector field which preserves ∇ (for it preserves all k-jets of ∇). The isotropy algebra $\mathcal{I}$ contains a copy of the Lie algebra of $SL(2,\mathbb{R})$: a contradiction, since $\mathcal{I}$ is of dimension two. $\square$

Our proof of theorem 1.3 will need analyticity in another essential way. We will make use of an extendability result for local Killing fields proved first for Nomizu in the Riemannian setting [30] and generalized then for rigid geometric structures by Amores and Gromov [2, 18] (see also [11, 13, 16]). This phenomenon states roughly that a local Killing field of a *rigid analytic* geometric structure can be extended along any curve in M. We then get a multivalued Killing field defined on all of M or, equivalently, a global Killing field defined on the universal cover. In particular, the Killing algebra in the neighborhood of any point is the same (as long as M is connected).

Recall here the following theorem which is an immediate consequence of Nomizu-Amores-Gromov's results:

Theorem 4.5. *Let M be a compact simply connected real analytic manifold admitting a real analytic locally homogeneous rigid geometric structure.*

Then M is isomorphic to a homogeneous space G/I endowed with a G-invariant geometric structure.

Proof. Since ϕ is locally homogeneous and M is simply connected and compact, the local transitive action of the Killing algebra extends to a global action of the associated simply connected Lie group G (we need compactness to ensure that vector fields on M are complete). All orbits have to be open, so there is only one orbit: the action is transitive and M is a homogeneous space. □

Let's go back now to the proof of theorem 1.3. As before, in the real analytic setting Gromov's stratification theorem shows that the locally homogeneous open dense set has to be dense [18, 13]. Note also that Nomizu's extension phenomenon doesn't imply that the extension of a family of pointwise linearly independent Killing fields, stays linearly independent. In general, the extension of a localy transitive Killing algebra, fails to be transitive on a nowhere dense analytic subset S in M. The unimodular affine connection is locally homogeneous on each connected component of $M \setminus S$.

We now prove that S is empty.

Assume by contradiction that S is not empty. Then we have the following crucial:

Lemma 4.6. *(i) The Killing algebra $\mathfrak{g}$ of ∇ has dimension two and the isotropy algebra at a point of S is one dimensional.*
(ii) $\mathfrak{g}$ is isomorphic to the Lie algebra of the affine group of the line.

Proof. (i) Since the Killing algebra admits a nontrivial open orbit in M, its dimension is ≥ 2. Pick up a point $s \in S$ and consider the linear morphism $ev(s) : \mathfrak{g} \to T_sM$ which associates to an element $K \in \mathfrak{g}$ its value $K(s)$. The kernel of this morphism is the isotropy $\mathcal{I}$ at s. Since the $\mathfrak{g}$-action is nontransitive in the neighborhood of s, the range of $ev(s)$ is dimension ≤ 1. This implies that the isotropy at s is of dimension at least dim $\mathfrak{g} - 1$.

Assume, by contradiction, that the Killing algebra has dimension at least three. Then the isotropy at s is of dimension at least two. By lemma 4.4, this dimension never equals two. Consequently, the isotropy algebra at $s \in S$ is three dimensional. The isotropy algebra contains then a copy of the Lie algebra of $SL(2,\mathbb{R})$ (see the proof of lemma 4.4).

The local action of $SL(2,\mathbb{R})$ in the neighborhood of s is conjugated to the the standard linear action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$. This action has two orbits: the point s and $\mathbb{R}^2 \setminus \{s\}$. The open orbit $\mathbb{R}^2 \setminus \{s\}$ identifies with a homogeneous space $SL(2,\mathbb{R})/I$. Precisely, the stabilizer I in $G = SL(2,\mathbb{R})$ of a nonzero vector $x \in T_sM$ is conjugated to the following one parameter unipotent subgroup of $SL(2,\mathbb{R})$: $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, with $b \in \mathbb{R}$. The action of $SL(2,\mathbb{R})$ on $SL(2,\mathbb{R})/I$ preserves the induced flat torsion free affine connection coming from $\mathbb{R}^2$.

By proposition 8 in [3], the only $SL(2,\mathbb{R})$-invariant affine connection on $SL(2,\mathbb{R})/I$ is the previous flat torsion free connection. Another way to prove this result is to consider the difference of a $SL(2,\mathbb{R})$-invariant connection with the standard one. We get a $(2,1)$-tensor on $SL(2,\mathbb{R})/I$ which is $SL(2,\mathbb{R})$-invariant. Equivalently, we get a $ad(I)$-invariant $(2,1)$-tensor on the quotient of the Lie algebra $sl(2,\mathbb{R})$ by the infinitesimal generator of I [1]. A straightforward computation shows that the tensor has to be trivial. This gives a different proof of the uniqueness of a $SL(2,\mathbb{R})$-invariant connection (a different proof can be find in [33], lemma 5.9).

By analyticity, ∇ is torsion free and flat on all of M. In particular, ∇ is locally homogeneous on all of M. This is in contradiction with our assumption.

(ii) The Killing algebra is two dimensional. Thus it coincides with $\mathbb{R}^2$ or with the Lie algebra of the affine group. Consider K_1, K_2 a basis of the local Killing algebra and extend K_1, K_2 along a topological disk with $s \in S$ in its boundary.

Recall that ∇ is unimodular and let vol be the volume form associated to ∇.

In the case where the Lie algebra is $\mathbb{R}^2$, $vol(K_1, K_2)$ is a nonzero constant (for being invariant by the Killing algebra and thus constant on the locally homogenous open set). Hence the Lie algebra acts transitively in the neighborhood of $s \in S$. This is a contradiction: S is then empty and ϕ is locally homogeneous on all of M. $\square$

For the sequel, let K_1, K_2 be two local Killing fields at $s \in S$ which span the Killing algebra. We assume, without loss of generality, that K_1 and K_2 verify the Lie bracket relation $[K_1, K_2] = K_1$.

Recall that K_1, K_2 don't vanish both at a point $s \in S$. Indeed, if not the isotropy at s has dimension 2, which is impossible by lemma 4.4.

Notice that $vol(K_1, K_2)$ is not invariant by the action of the Killing algebra (since the adjoint representation of $Aff(\mathbb{R})$ is nontrivial). But we still have:

Proposition 4.7. *The local function $vol(K_1, K_2)$ is constant on the orbits of the flow generated by K_1.*

Proof. The adjoint action $ad(K_1)$ on $\mathfrak{g}$ is nilpotent. In particular, the adjoint representation of the one parameter subgroup generated by K_1 is unimodular. Consequenlty, the local flow generated by K_1 preserves $vol(K_1, K_2)$. $\square$

Proposition 4.8. *S is a smooth 1-dimensional manifold (diffeomorphic to a finite union of circles).*

Proof. By lemma 4.6, the isotropy $\mathcal{I}$ at a chosen point $s \in S$ is one dimensional and the range of the map $ev(s)$ is dimension one. In particular, the orbit of s under the action of $\mathfrak{g}$ is one dimensional. The image of $ev(s)$ coincides with T_sS. Consequently, the $\mathcal{I}$-action on T_sM preserves the line T_sS.

This shows that $\mathfrak{g}$ acts transitively on each connected component of S.

In particular, each connected component of S is a one dimensional smooth submanifold in M (recall that S is nowhere dense in M).

Since M is compact, each connected component of S is diffeomorphic to a circle.

$\square$

We also have:

Proposition 4.9. *(i) Up to a finite cover, each connected component of $M \setminus S$ is locally modelled on the affine group of the line endowed with a translation invariant connection.*

(ii) $M \setminus S$ admits a (nonsingular) $\mathfrak{g}$-invariant foliation $\mathcal{F}_1$ by lines.

(iii) $\mathcal{F}_1$ coincides with the kernel of a (nonsingular) closed $\mathfrak{g}$-invariant one form ω.

Proof. (i) Since the $\mathfrak{g}$-action on $M \setminus S$ is simply transitive, ∇ is locally isomorphic to a translation-invariant connection ∇_0 on $Aff(\mathbb{R})$. It follows that each connected component of $M \setminus S$ is locally modelled on G/Λ, where G is the automorphism group of ∇_0 and Λ is the stabilizer of the identity. The identity component of G is $Aff(\mathbb{R})$, thus Λ is a discrete group. Moreover, Λ is an algebraic group since it is isomorphic to the stabilizer in $SL(2, \mathbb{R})$ of the k-jet of ∇_0 for $k \in \mathbb{N}$ large enough (see the proof of lemma 4.4).

Consequently, Λ is a finite group and G has a finite number of connected components. Up to a finite cover, we can assume that $M \setminus S$ is locally modelled on the affine group.

(ii) The Lie bracket relation $[K_1, K_2] = K_1$ implies that the flow of K_2 normalizes the flow of K_1 and thus the foliation spaned by K_1 is K_2-invariant. Consequently, the foliation spaned by K_1 is $\mathfrak{g}$-invariant and it defines a foliation $\mathcal{F}_1$ well defined on $M \setminus S$ (it comes from a translation-invariant foliation on $Aff(\mathbb{R})$).

(iii) We locally define the one form ω such that $\omega(K_1) = 0$ and $\omega(K_2) = 1$. Since the basis K_1, K_2 is well defined, up to a Lie algebra isomorphism (which necessarily preserves the derivative algebra $\mathbb{R}K_1$ and sends K_2 on $K_2 + \beta K_1$, for $\beta \in \mathbb{R}$), ω is globally defined on $M \setminus S$. Also by Lie-Cartan's formula [1] $d\omega(K_1, K_2) = -\omega([K_1, K_2]) = 0$.

Thus the foliation is transversaly Riemannian in the sense of [27]. Another way to see it, is to observe that the projection of the local Killing field $K_2 + \beta K_1$ on $TM/T\mathcal{F}_1$ is well defined (it doesn't depend on β). It defines a transverse vector field $\tilde{K}_2$ and, consequently, $\mathcal{F}_1$ is transversaly Riemannian. □

We prove now:

Proposition 4.10. *$\mathcal{F}_1$ extends to a nonsingular foliation on all of M.*

We study the local situation in the neighborhood of $s \in S$. For this local analysis we will also denote by S the connected component of s in S.

Denote by $Y \in \mathfrak{g}$ a generator of $\mathcal{I}$ and by $X \in \mathfrak{g}$ a Killing field such that $X(s)$ span T_sS. By applying an automorphism of the Lie algebra of the affine group we can assume that either $Y = K_1$ and $X = K_2$, or $Y = K_2$ and $X = K_1$.

Proof. **Case I: unipotent isotropy**

Assume first that $Y = K_1$ and $X = K_2$.

Because of the Lie bracket relation, the isotropy $\mathcal{I}$ at s acts trivially on T_sS. This implies that its generator Y acts trivially on the unique geodesic passing through s and tangent to the direction T_sS. Consequently Y vanishes on this geodesic which has to coincide locally to S (for S is the subset of M where $\mathfrak{g}$ doesn't act freely).

Since the Y-action on T_sM is trivial on T_sS and volume preserving, it is unipotent. In local exponential coordinates (x, y) at s, Y is linearized and so conjugated to the linear vector field $y\frac{\partial}{\partial x}$. In these coordinates S identifies locally with $y = 0$ and the time t of the flow generated by Y is $(x, y) \to (x + ty, y)$.

Notice that the flow of Y preserves a unique foliation in the neighborhood of s, which is given by $dy = 0$.

Case II: semi-simple isotropy

We assume now that $Y = K_2$ and $X = K_1$.

In the neighborhood of $s \in S$ the local foliation spaned by the nonsingular vector field K_1 agrees with $\mathcal{F}_1$ on $M \setminus S$. Since this stands in the neigborhood of each point of S, the foliation $\mathcal{F}_1$ extends to a nonsingular foliation on M by adding the leaf S.

If S admits several connected components, the previous argument applies in the neighborhood of each component of S. □

Lemma 4.11. *(i) M is homeomorphic to a torus.*

(ii) Each connected component of $M \setminus S$ is homeomorphic to a cylinder $\mathbb{R} \times S^1$.

Proof. (i) It follows from Poincaré-Hopf's theorem, since M admits a nonsingular foliation.

(ii) We prove first that $M \setminus S$ doesn't admit any connected component homeomorphic to a disk. Assume, by contradiction, that there is a connected component homeomorphic to a disk D. By Amores-Nomizu's extension result, the Killing fields K_1 and K_2 extend on all of D (and even on the closure of D in M). Since they are tangent to the border (they preserve D and each connected component of S), they are complete. This complete action of $Aff(\mathbb{R})$ is transitive and D identifies with a quotient of $Aff(\mathbb{R})$ by a discrete subgroup Γ. Since the $Aff(\mathbb{R})$-action preserves the finite volume of D given by vol, Γ has to be a lattice in $Aff(\mathbb{R})$: a contradiction.

Each connected component of $M \setminus S$ is an open oriented surface with nonpositive Euler characteristic class. Since the union of all connected components has a Euler characteristic $\chi(M) = 0$, it follows that each connected component has a vanishing Euler class and hence it is homeomorphic to a cylinder. □

By proposition 4.9, each connected component of $M \setminus S$ is locally modelled on the affine group. The corresponding holonomy morphism is a homomorphism $\mathbb{Z} \to Aff(\mathbb{R})$. Up to a inner automorphism of the affine group we can assume that the image lies either in the one parameter group generated by

K_1 (case of unipotent holonomy), or in the one parameter group generated by K_2 (case of semi-simple holonomy).

We deal first with the case where there exists one connected component M_0 of $M \setminus S$ with unipotent holonomy.

Case I: unipotent holonomy

We give a more precise description of $\mathcal{F}_1$ on M_0.

Proposition 4.12. *(i) The Killing field K_1 extends on all of M_0.*

(ii) The function $vol(K_1, K_2)$ extends on all of M_0 in a $\mathcal{F}_1$-invariant function.

(iii) The leaves of $\mathcal{F}_1$ are closed in M_0.

(iv) $\mathcal{F}_1$ is the trivial foliation by circles of the cylinder M_0.

Proof. (i) Since K_1 is invariant under the action of the holonomy group, K_1 is well-defined on M_0.

(ii) We have seen that $vol(K_1, K_2)$ is $ad(K_1)$-invariant. In particular, this function is invariant under the action of the holonomy group: it extends on all of M_0 in a function which is constant on the K_1-orbits.

(iii) The level sets of $vol(K_1, K_2)$ are compact, since $vol(K_1, K_2)$ doesn't vanish on M_0 (K_1 and K_2 being pointwise linearly independent on $M \setminus S$), but vanishes on ∂M_0. Since each orbit of K_1 lies in a level set of $vol(K_1, K_2)$ and K_1 is complete in restriction to this compact set, the K_1-orbits are exactly the level sets of the function $vol(K_1, K_2)$ (notice also that K_1 is nonsingular on M_0).

(iv) Let $exp(sK_1)$, with $s \in \mathbb{R}$, be the generator of the image of the holonomy morphism. Then the flow of K_1 is parametrized by the circle $\mathbb{R}/s\mathbb{Z}$ and any orbit of K_1 is a loop homotopic to the generator of the fundamental group of M_0. By continuity, the two components of S boarding M_0 are also K_1-orbits parametrized by $\mathbb{R}/s\mathbb{Z}$.

Moreover the transversal F of the foliation $\mathcal{F}_1$ is Hausdorff: this is a consequence of (iii) and of the fact that $\mathcal{F}_1$ is transversaly Riemannian by proposition 4.9 (see [27], chapter 3, Proposition 3.7).

Recall that F is parametrized by the flow of $\tilde{K}_2$ (projection of K_2 on $TM/T\mathcal{F}_1$) which is globaly defined on M_0. More precisely, F is diffeomorphic to the maximal domain of definition $]a, b[$ of a integral curve of $\tilde{K}_2$.

□

The function $vol(K_1, K_2)$ is a well-defined function on the space of leaves F of the foliation $\mathcal{F}_1$. Since $vol(K_1, K_2)$ vanishes on S, this function has limits equal to 0 in both a and b. Thus it must admit at least one global extremum on F.

The following proposition gives a contradiction and achieves this case:

Proposition 4.13. *The function $vol(K_1, K_2)$ doesn't admit local extrema on F.*

Proof. The action of the local Killing field K_2 preserves the volume of ∇. If t is the local time of a local K_2-orbit of a point in M_0, the function $vol(e^t K_1, K_2)$ must be constant on the K_2-orbit. It follows that $vol(K_1, K_2)$

is strictly decreasing with respect to the local parameter t, which was also seen to be a local coordinate on F. $\square$

Case II: semi-simple holonomy

We adress here the remaining case where all connected components of $M \setminus S$ have a semi-simple holonomy. Let M_0 be a connected component of $M \setminus S$. Up to conjugacy, the image of the holonomy morphism lies in the one parameter subgroup generated by K_2. It follows that K_2 is invariant under the holonomy morphisme and gives a nonsingular Killing vector field well-defined on M_0. This vector field is transverse to the foliation $\mathcal{F}_1$ generated by K_1.

Proposition 4.14. *(i) The orbits of K_2 are closed.*

(ii) All connected components of S are orbits of the vector field K_2.

(iii) K_2 extends to a Killing field defined on all of M, all of whose orbits are closed.

Proof. Consider a connected component M' of $M \setminus S$ and let $exp(sK_2)$, with $s \in \mathbb{R}$, be the generator of the image of the holonomy morphism.

(i) The flow of K_2 is parametrized by the circle $\mathbb{R}/s\mathbb{Z}$. In particular, all the K_2-orbits are closed.

(ii) By continuity, the two connected components of S boarding M' are closed orbits parametrized by $\mathbb{R}/s\mathbb{Z}$.

iii) The K_2-orbits generate the trivial foliation by circles of the cylinder M_0. In particular, any K_2-orbit generates (as a loop) the fundamental group of M'. The vector field K_2 can be extended on an open neighborhood of M_0 in M containing the closure of M_0 (since the monodromy of K_2 is trivial along the generator of the fundamental group). Thus all connected components of $M \setminus S$ have a holonomy group generated by $exp(sK_2)$ and K_2 is a globally defined Killing field on M with orbits parametrized by $\mathbb{R}/s\mathbb{Z}$.

Lemma 4.15. *There exists global coordinates (x, y) on the universal cover of M with respect to which the expressions of the pull-back of K_1 and K_2 are :*

$\tilde{K_2} = \frac{\partial}{\partial x}$ and $\tilde{K}_1 = e^{-x}(f'(y)\frac{\partial}{\partial x} + f(y)\frac{\partial}{\partial y})$, where f is an analytic function with a simple zero in 0 and $\frac{f'}{f}$ is periodic.

Proof. Let x be the time coordinate of the K_2-flow. Let F be a circle in M which is a global transversal to the the foliation generated by K_2. Let y be a coordinate on the universal cover of F such that $vol(K_2, \cdot) = dy$ and $y = 0$ on a component of S. Then (x, y) are global coordinates on the universal cover of M such that $\tilde{K_2} = \frac{\partial}{\partial x}$ and the pull-back of the volume form is $dx \wedge dy$. The closed loop F acts on $\mathbb{R}^2$ by deck transformation which is a y-translation.

Since $[\tilde{K_1}, \tilde{K_2}] = \tilde{K_1}$ and the divergence of $\tilde{K_1}$ with respect to $dx \wedge dy$ vanishes, a direct computation shows that $\tilde{K_1} = e^{-x}(f'(y)\frac{\partial}{\partial x} + f(y)\frac{\partial}{\partial y})$, where f an analytic function of the coordinate y.

The foliation $\mathcal{F}_1$ generated by K_1 being well-defined on M (by proposition 4.10), the slope $\frac{f'}{f}$ of K_1 is periodic (since it is invariant by deck transformations). On the other hand, K_1 and K_2 have to be colinear on S and consequently, f vanishes on S. The Lie bracket relation implies $f'(0) \neq 0$. □

Proposition 4.16. *The pull-back $\tilde{\nabla}$ of ∇ on M admits constant Christoffel symbols with respect to the vector fields $A = -\frac{f'}{f}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and $B = (1 - \frac{f'}{f}y)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. In particular, there exists constants $a, b \in \mathbb{R}$ such that $\tilde{\nabla}_B B = aA + bB$.*

Proof. The vector fields A and B commute with both $\tilde{K}_1$ and $\tilde{K}_2$. They are linearly independent and analytic on the open set where the action is transitive, and since $\tilde{K}_1$ and $\tilde{K}_2$ preserve $\tilde{\nabla}$, the Christoffel symbols with respect to A and B have to be constant. □

Proposition 4.17. *Let Γ^k_{ij} be the Christoffel symbols of $\tilde{\nabla}$ with respect to the coordinates (x, y). Then Γ^k_{ij} are periodic functions of y.*

Proof. Since $\frac{\partial}{\partial x}$ preserves $\tilde{\nabla}$, the analytic functions Γ^k_{ij} depend only on y. The closed loop F in M acts on the universal cover by y-translation. This action preserves $\tilde{\nabla}$ and thus Γ^k_{ij} are periodic. □

From proposition 4.16 we have

$$aA + bB = \tilde{\nabla}_B B = \tilde{\nabla}_{(1-\frac{f'}{f}y)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}}(1 - \frac{f'}{f}y)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

We identify the coefficients of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ in the previous equality. This leads to

$$1)\ a(-\frac{f'}{f})+b(1-\frac{f'}{f}y) = (1-\frac{f'}{f}y)^2\Gamma^1_{11}+(1-\frac{f'}{f}y)y\Gamma^1_{12}-y^2(\frac{f'}{f})'+y^2\Gamma^1_{22}+(-\frac{f'}{f})y$$

and to

$$2)\ a + by = (1 - \frac{f'}{f}y)^2\Gamma^2_{11} + (1 - \frac{f'}{f}y)y\Gamma^2_{12} + (1 - \frac{f'}{f}y)y\Gamma^2_{21} + y + y^2\Gamma^2_{22}.$$

The function y verifies then two equations of second order with periodic coefficients (since Γ^k_{ij} and $\frac{f'}{f}$ have a same period). But y is not periodic and consequently, both equations of second order have to be trivial.

We will see that this leads to a contradiction.

First remark that the vanishing of the constant term in 1) leads to:

$$a(-\frac{f'}{f}) + b = \Gamma^1_{11}.$$

Since $\frac{f'}{f}$ admits a pole in 0 and Γ^1_{11} is analytic, this implies $a = 0$ and $\Gamma^1_{11} = b \in \mathbb{R}$.

The constant term in 2) gives then

$$\Gamma^2_{11} = a = 0.$$

The vanishing of the coefficient of y in 2) gives then:

$$-2\frac{f'}{f}\Gamma^2_{11} + (\Gamma^2_{12} + \Gamma^2_{21}) + 1 = b.$$

Consequently,

$$\Gamma^2_{12} + \Gamma^2_{21} = b - 1 \in \mathbb{R}.$$

The coefficient of y^2 in 2) is:

$$\Gamma^2_{22} - \frac{f'}{f}(\Gamma^2_{12} + \Gamma^2_{21}) = 0.$$

This yields $\Gamma^2_{22} = 0$ and $\Gamma^2_{12} + \Gamma^2_{21} = b - 1 = 0$.

The vanishing of the coefficient of y in 1) yields a contradiction. Indeed

$$b(-\frac{f'}{f}) = -2\frac{f'}{f}\Gamma^1_{11} + \Gamma^1_{12} - \frac{f'}{f},$$

with $b = \Gamma^1_{11} = 1$ leads to

$$-2\frac{f'}{f} + \Gamma^2_{12} = 0,$$

a contradiction since Γ^1_{12} is analytic. This achieves the proof of theorem 1.3. $\square$

I would like to thank Adolfo Guillot who gave me the following example of non locally homogeneous unimodular analytic affine connection:

Proposition 4.18. *Let ∇ be a connection on $\mathbb{R}^2$ such that $\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = \mu y^3 \frac{\partial}{\partial x}$ and all other Chrsitoffel symbols vanish. Then the (torsion-free) unimodular affine connection determined by ∇ and the volume form $vol = dx \wedge dy$ is locally homogeneous on $y > 0$ and on $y < 0$, but not on all of $\mathbb{R}^2$.*

Proof. We check easily that ∇ and the volume form are invariant by the flows of $\frac{\partial}{\partial x}$ and of $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$.

Remark that the invariant vector fields of the previous action are $A = y\frac{\partial}{\partial y}$ and $B = \frac{1}{y}\frac{\partial}{\partial x}$. They are of constant volume and ∇ has constant Christoffel symbols with respect to A and B:

$$\nabla_A A = A, \nabla_B B = \mu A, \nabla_B A = 0, \nabla_A B = [A, B] = -B.$$

Thus the unimodular affine connection is locally homogeneous on the open sets $y > 0$ and $y < 0$, where A and B are pointwise linearly independent.

The only nonzero component of the curvature tensor is

$$R(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\frac{\partial}{\partial x} = \nabla_{\frac{\partial}{\partial x}}\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x} - \nabla_{\frac{\partial}{\partial y}}\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = -3\mu y^2\frac{\partial}{\partial x}.$$

The curvature tensor vanishes exactly on $y = 0$. Thus ∇ is not locally homogeneous on all of $\mathbb{R}^2$. $\square$

5. Locally homogeneous affine connections

Recall that T. Nagano and K. Yagi completely classified torsion free flat affine connections (affine structures) on the real two torus [29]. They proved that, except a well described family of incomplete affine structures (see the nice description given in [4]), all the other are homogeneous constructed from faithful affine actions of $\mathbb{R}^2$ on the affine plane $GL(2,\mathbb{R}) \ltimes \mathbb{R}^2/GL(2,\mathbb{R})$ admitting an open orbit. This induces a translation invariant affine structure on $\mathbb{R}^2$. The quotient of $\mathbb{R}^2$ by a lattice is a homogeneous affine two dimensional tori.

Notice that the previous homogeneous affine structures are complete if and only if the corresponding $\mathbb{R}^2$-actions on the affine plane are transitive.

In particular, Nagano-Yagi's result prove that a real two dimensional torus locally modelled on the affine space $GL(2,\mathbb{R}) \ltimes \mathbb{R}^2/GL(2,\mathbb{R})$ such that the linear part of the holonomy morphism lies in $SL(2,\mathbb{R})$ is always complete and homogeneous (the corresponding torsion free flat affine connection on the torus is translation invariant).

Here we prove:

Theorem 5.1. *Let M be a compact surface locally modelled on a homogeneous space G/I such that G/I admits a G-invariant affine connection ∇ and a volume form. Then:*

(i) the corresponding affine connection on M is homogeneous (up to a finite cover), except if ∇ is the Levi-Civita connection of a hyperbolic metric (and the genus of M is ≥ 2).

(ii) the corresponding affine connection on M is complete.

Remark 5.2. *In particular, a unimodular locally homogeneous affine connection on the two torus has constant Christoffel symbols with respect to global translation invariant coordinates.*

Indeed, by Gauss-Bonnet's theorem, a two torus doesn't admit hyperbolic metrics. Thus theorem 5.1 implies theorem 1.4.

Proof. Note first that, in the case where I is compact, theorem 3.2 proves that M is complete. Moreover, if M is of genus 0 then the underlying locally homogeneous Riemannian metric has to be of positive sectional curvature (by Gauss-Bonnet's theorem) and G/I coincides with the standard sphere S^2 seen as a homogeneous space of S^3. Since M is simply connected, the developing map gives a global isomorphism with S^2.

Also if the genus of M is one, the underlying locally homogeneous metric on M has to be flat. It follows that G/I coincides with the only homogeneous simply connected flat Riemannian space $O(2,\mathbb{R}) \ltimes \mathbb{R}^2/O(2,\mathbb{R})$. Bieberbach's theorem (see, for instance, [35]) implies then that M is homogeneous (up to a finite cover).

Surfaces of genus $g \geq 2$ cannot admit a homogeneous geometric structure locally modelled on G/I, with I compact. Indeed, here the Riemannian metric induced on M is of constant negative curvature. It is well know that the isometry group of a hyperbolic metric on M is finite (hence it cannot act transitively) [20].

Consider now the case where I is noncompact.

Denote also by ∇ the locally homogeneous affine connection induced on M. The Killing algebra $\mathfrak{g}$ acts locally on M preserving ∇ and a volume form vol. Assume ∇ is not the torsion free flat affine connection. The proof of theorem 1.3 implies that $\mathfrak{g}$ is of dimension at most 3. Therefore G is of dimension 3 and I is a one parameter subgroup in G. Since I is supposed noncompact, its (faithful) isotropy action on T_oG/I, where o is the origin point of G/I, identifies I either with a semi-simple, or with a unipotent one parameter subgroup in $SL(2,\mathbb{R})$.

Case I: semi-simple isotropy

The isotropy action on T_oG/I preserves two line fields. Consequently, there exists on G/I two G-invariant line fields. Since G/I admits also a G-invariant volume form, there exists on G/I a G-invariant Lorentz metric q.

By Poincaré-Hopf's theorem M is a torus and q has to be flat (by Gauss-Bonnet's theorem). It follows that G is the automorphism group SOL of the flat Lorentz metric $dxdy$ on $\mathbb{R}^2$. By the Lorentzian version of Bieberbach's theorem, this structure is known to be complete and homogeneous (up to a finite cover of M) [12].

Case II: unipotent isotropy

Here the isotropy action preserves a vector field in T_oG/I. This yields a G-invariant vector field $\tilde{X}$ on G/I and a G-invariant one form $\tilde{\omega} = vol(\tilde{X}, \cdot)$. We then have:

Proposition 5.3. *M admits a $\mathfrak{g}$-invariant vector field X and a $\mathfrak{g}$-invariant closed one form ω vanishing on X. Thus the foliation $\mathcal{F}$ generated by X is transversally Riemannian.*

Proof. Since $d\tilde{\omega}$ and vol are G-invariant, there exists $\lambda \in \mathbb{R}$ such that $d\tilde{\omega} = \lambda vol$.

Denote by ω the one form on M associated to ω and by X the vector field induced on M by $\tilde{X}$. It follows that ω vanishes on X. Also $\int_M d\omega = \lambda vol(M)$, where the volume of M is calculated with respect to the volume form induced by vol. Stokes' theorem yield $\lambda = 0$ and, consequently, ω is closed. □

Proposition 5.4. *The normal subgroup H of G which preserves each leaf of the foliation $\tilde{\mathcal{F}}$ generated by $\tilde{X}$ on G/I is two dimensional and abelian.*

Proof. The group G preserves the foliation $\tilde{\mathcal{F}}$ and its transverse Riemannian structure. Therefore an element of G fixing one leaf of $\tilde{\mathcal{F}}$, will fixe all the leaves. The subgroup H is nontrivial, since it contains the isotropy. The action of G/H on the transversal of the foliation preserves a Riemannian structure, so it is of dimension at most one. Since this action has to be transitive, the dimension of G/H is exactly one and the dimension of H is two.

Since H preserves $\tilde{X}$, the elements of H commutes in restriction to each leaf of $\tilde{\mathcal{F}}$. Hence H is abelian. □

We will make use of the following result proved in [14] (pages 17-19):

Lemma 5.5. *$\tilde{X}$ is a central element in $\mathfrak{g}$. Consequently, X is a global Killing field on M preserved by $\mathfrak{g}$.*

In order to prove that M is homogeneous, we will construct a second global Killing field on M which is not tangent to the foliation $\mathcal{F}$ generated by X. For this we will study the holonomy group.

Once again M is of genus one, since it admits a nonsingular vector field. Since the fundamental group of the torus is abelian, its image Γ by the holonomy morphism is an abelian subgroup of G.

We prove first that Γ is nontrivial. Assume by contradiction that Γ is trivial. Then the developing map is well defined on M and we get a local diffeomorphism $dev : M \to G/I$. The image has to be open and closed (for M is compact). Recall that G is connected and hence dev is surjective. By Ehresmann's submersion theorem, dev is a covering map. Remark 2.6 shows that G/I can be considered simply connected, which implies dev is a diffeomorphism: a contradiction, since M is not simply connected.

Considering finite covers of M, the previous proof rules out the case where Γ is finite. Consequently, Γ is an infinite subgroup of G.

Consider its (real) Zariski closure $\overline{\Gamma}$ in G, which is an abelian subgroup of positive dimension. This is possible, since by Lie's classification [28, 31], G is locally isomorphic to a real algebraic group acting (transitively) on G/I. Therefore, up to a finite cover, we can assume that G is algebraic and G/I is still simply connected.

Up to a finite cover of M, we can assume $\overline{\Gamma}$ connected (algebraic groups admit at most finitely many connected components).

Proposition 5.6. *Any element a of the Lie algebra of $\overline{\Gamma}$ defines a global Killing field on M.*

Moreover, if the one parameter subgroup of G generated by a intersects Γ nontrivially, then the orbits of the corresponding Killig field on M are closed.

Proof. The action of Γ on a (by adjoint representation) being trivial, a defines a Γ-invariant vector field on G/I which descends on M.

Assume now that Γ intersects nontrivially the one parameter subgroup generated by a. One orbit of the corresponding Killing field on M develops in the model G/I as one orbit of the Killing field a. Since vector fields on compact manifolds are complete, the image of the developing map contains all of the orbit of a. In particular, the corresponding orbit of a contains distinct points which are in the same Γ-orbit. Therefore, the corresponding orbit on M is closed. □

We prove now:

Proposition 5.7. *$\overline{\Gamma}$ is not a subgroup of H.*

Proof. Consider a one parameter subgroup in G, generated by an element in the Lie algebra of $\overline{\Gamma}$, which nontrivially intersects Γ. By proposition 5.6, a defines a global Killing field K with closed orbits on M.

Assume by contradiction that $\overline{\Gamma}$ is a subgroup of H. It follows that the orbits of K coincide with those of X. Consequently, the orbits of X are closed and the space of leaves of $\mathcal{F}$ is a one dimensional manifold (see proposition 4.9, point (v)). Since M is compact, the space of leaves is diffeomorphic to a circle S^1.

Consider the developing map $dev : \tilde{M} \to G/I$ of the G/I-structure. In particular, this is also the developing map of the transverse structure of the foliation $\mathcal{F}$. Since the holonomy group acts trivially on the transversal $\tilde{T}$ of $\tilde{\mathcal{F}}$, dev descends to a local diffeomorphism from the space of leaves of $\mathcal{F}$ (parametrized by S^1) to $\tilde{T}$.

Since G/I is simply connected, the closed one form $\tilde{\omega}$ admits a primitive $\tilde{f} : G/I \to \mathbb{R}$. Consequently, $\tilde{f}$ is a first integral for $\tilde{\mathcal{F}}$ and $\tilde{f} \circ dev$ descends to a local diffeomorphism from S^1 to $\mathbb{R}$. This map has to be onto since the image is open and closed. We get a topological contradiction. $\square$

By proposition 5.6, any one parameter subgroup in G generated by an element of the Lie algebra of $\overline{\Gamma}$ non contained in the Lie algebra of H provides a global Killing field K on M such that the abelian group generated by the flows of X and of K acts transitively on M. Therefore the G/I-structure on M is homogeneous.

(ii) Consider $\tilde{X}, \tilde{K}$ the corresponding Killing vector fields on G/I. They generate a two dimensional abelian subgroup A of G acting with an open orbit on G/I. Recall that $vol(\tilde{X}, \tilde{K})$ is an A-invariant function on G/I. Hence, $vol(\tilde{X}, \tilde{K})$ is a nonzero constant in restriction to the open orbit. By continuity, $vol(\tilde{X}, \tilde{K})$ equals the same nonzero constant on the closure of the open orbit. This implies that $\tilde{X}, \tilde{K}$ remain linearly independent on the closure of the open orbit, which implies that the open orbit is also closed. Since G/I is connected, the open orbit is all of G/I. $\square$

The previous proof combined with a result of [14] leads to the following classification result:

Theorem 5.8. *Let M be a compact surface locally modelled on a homogeneous space G/I such that G/I admits a G-invariant affine connection ∇ and a volume form.*

If I is noncompact, then either ∇ is torsion free and flat and $G = SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$, or G is three dimensional and we have the following possibilities:

(i) either G is the unimodular SOL group and its action preserves a flat (two dimensional) Lorentz metric;

(ii) or G is the Heisenberg group and its action preserves the standard flat torsion free affine connection on $\mathbb{R}^2$ together with a nonsingular closed one form $\tilde{\omega}$ and with a nontrivial parallel vector field $\tilde{X}$ such that $\tilde{\omega}(\tilde{X}) = 0$;

(iii) or G is isomorphic to the product $\mathbb{R} \times Aff(\mathbb{R})$ and its action preserves the canonical bi-invariant torsion free and complete connection of $Aff(\mathbb{R})$ (for which the full automorphism group is $Aff(\mathbb{R}) \times Aff(\mathbb{R})$) together with a geodesic Killing field.

The model (i) was obtained in the previous proof in the case where the isotropy is semi-simple. The models (ii) and (iii) correspond to the case where the isotropy is unipotent. This classification follows from [14] (see Proposition 3.5 and pages 15-19).

Theorem 5.1 can be deduced from theorem 5.8. Indeed, in the case (ii) G preserves a flat torsion free affine connection and Nagano-Yagi's result applies. The completeness of compact surfaces locally modelled on the homogeneous space (iii) was proved in Proposition 9.3 of [36] (the homogeneity follows from the proof of Proposition 10.1 in [36]).

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